Implicative Quasi-semi-lattices

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Introduction

As is well-known that semi-lattices are always present in the study of algebras related to fuzzy logics. Recall that a partially ordered set $(S, \leq)$ in which any pair of elements x and y of S has an infimum $x \wedge y$ is called a semi-lattice, thus a semi-lattice can be regarded as the generalization of a lattice [1]. It is easily seen that the definition of semi-lattice is equivalent to the conditions hold in $(S, \leq)$: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $x \wedge y = y \wedge x$ and $x \wedge x = x$ for any $x, y, z \epsilon S$, so semi-lattices can also be seen as the commutative semigroups of idempotents and often appear in the pure algebraic investigation. Implicative semi-lattices which is an important subclass of semi-lattices were studied by Nemitz in [2]. Subsequently, weak implicative semi-lattices were introduced and the congruences on a weak implicative semi-lattice were characterized in [3].

On the other hand, Chajda introduced the notion of q-lattices as a generalization of lattices in 1993 [4]. Moreover, the ideals and filters of a q-lattice have been discussed in [5]. In order to establish more relationship between fuzzy logics and quantum computational logics, we want to introduce implicative quasi-semi-lattices as a generalization of implicative semi-lattices similarly as q-lattices generalized lattices in this paper.

Implicative quasi-semi-lattices

➢ **Implicative quasi-semi-lattices**

A *quasi-semi-lattice with top* is a system $(L, \wedge, 1)$ in which L is a non-empty set, $\wedge$ is a binary relation in L such that for any $x, y, z \epsilon L$, $x \wedge y = y \wedge x$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \wedge (y \wedge y) = x \wedge y$ and 1 is an element of L such that for any $x \epsilon L$, $x \wedge 1 = x \wedge x$ and $1 \wedge 1 = 1$.

Let $(L, \wedge, 1)$ be a quasi-semi-lattice with top. For $x, y \epsilon L$, if there exists the largest regular element z such that $z \wedge x \leq y$, then z is called the residual of x relative to y and denoted by $x \to y$. If for any $x, y \epsilon L$, $x \to y$ always exists, then $(L, \wedge, 1)$ is called an *implicative quasi-semi-lattice* and usually denoted by $(L, \wedge, \to, 1)$.

Here are some propositions about an implicative quasi-semi-lattice $(L, \wedge, \to, 1)$ for any $x, y, z \epsilon L$:

(1) $x \wedge y \leq z$ if and only if $x \leq y \to z$;
(2) $y \leq x \to y$;
(3) $x \to x = 1$ and $x \wedge x = 1 \to x$;
(4) If $x \leq y$, then $y \to z \leq x \to z$ and $z \to x \leq z \to y$;
(5) $x \leq y$ if and only if $x \to y = 1$;
(6) $(x \to y) \wedge (y \to z) \leq x \to z$;
(7) $x \to (y \to z) = (x \wedge y) \to z$;
(8) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
(9) $x \wedge (x \to y) = x \wedge y$.

Filters and filter congruences

➢ **Filters**

Let $(L, \wedge, \to, 1)$ be an implicative quasi-semi-lattice. A non-empty subset F of L is called a *filter* of $(L, \wedge, \to, 1)$, if the following conditions are satisfied:

(F1) If $x, y \epsilon F$, then $x \wedge y \epsilon F$;
(F2) If $x \epsilon F$ and $y \epsilon L$ with $x \leq y$, then $y \epsilon F$.

➢ **Filter congruences**

Let $(L, \wedge, \to, 1)$ be an implicative quasi-semi-lattice and θ be a binary relation on $(L, \wedge, \to, 1)$. Then θ is called a *congruence* on $(L, \wedge, \to, 1)$, if the following conditions are satisfied:

(1) θ is an equivalence relation;
(2) If $< x, y > \epsilon \theta$ and $< a, b > \epsilon \theta$, then $< x \wedge a, y \wedge b > \epsilon \theta$;
(3) If $< x, y > \epsilon \theta$ and $< a, b > \epsilon \theta$, then $< x \to a, y \to b > \epsilon \theta$.

For any $x, y \epsilon L$, if $< x \wedge x, y \wedge y > \epsilon \theta$ implies $< x, y > \epsilon \theta$, then θ is called a *filter congruence* on $(L, \wedge, \to, 1)$.

➢ **The relation between filters and filter congruences**

1. Let $(L, \wedge, \to, 1)$ be an implicative quasi-semi-lattice and F be a filter of $(L, \wedge, \to, 1)$. Define a binary relation θ_F by $< x, y > \theta_F$ if and only if $x \wedge f = y \wedge f$ for some $f \epsilon F$. Then θ_F is a filter congruence on $(L, \wedge, \to, 1)$.

   Let $(L, \wedge, \to, 1)$ be an implicative quasi-semi-lattice and θ be a filter congruence on $(L, \wedge, \to, 1)$. Then $F_\theta = 1/\theta$ is a filter of $(L, \wedge, \to, 1)$.

   Let $(L, \wedge, \to, 1)$ be an implicative quasi-semi-lattice, F be a filter of $(L, \wedge, \to, 1)$ and θ be a filter congruence on $(L, \wedge, \to, 1)$. Then $F = F_{\theta_F}$ and $\theta = \theta_{F_\theta}$.
   Denote $F(L)$ the set of all filters of $(L, \wedge, \to, 1)$ and $Con(L)$ the set of all filter congruences on $(L, \wedge, \to, 1)$. Then $F(L)$ and $Con(L)$ are **one-to-one**.

2. Let $(L, \wedge, \to, 1)$ be an implicative quasi-semi-lattice and A, B be two sets of $(L, \wedge, \to, 1)$. Then the set formed by the common elements of set A and set B is called the intersection of set A and set B, denoted by $A \cap B$.

   Let $(L, \wedge, \to, 1)$ be an implicative quasi-semi-lattice. If F_1, F_2 are two filters of $(L, \wedge, \to, 1)$ and θ_1, θ_2 are two filter congruences on $(L, \wedge, \to, 1)$, define $F_1 \vee F_2 = [F_1 \cup F_2)$ and $\theta_1 \vee \theta_2 = [\theta_1 \cup \theta_2)$, then $(F(L), \cap, \vee)$ and $(Con(L), \cap, \vee)$ are lattices. Moreover, there exists an **isomorphism** between $(F(L), \cap, \vee)$ and $(Con(L), \cap, \vee)$.

References

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