Fuzzy Henstock-Δ-Integral on Time Scales and Its Application

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Introduction

In this paper, we define the fuzzy Henstock-Δ-integral and fuzzy Δ-derivative on time scales(or briefly FH-Δ-integral, Δ-derivative). Then, we give some convergence theorems for this kind of nonabsolute convergent integrals. Finally, we obtain an existence theorem of the global solutions for this kind of fuzzy generalized differential systems.

Methods

- time scales;
- Arzela-Ascoli theorem;
- equicontinuity of sequences.

Main Results

Definition 3.1 *Suppose* $\tilde{f}:[a,b]_{\mathbb{T}}\to\mathbb{R}_{\mathscr{F}}$ *and there exists* $\tilde{F}:[a,b]_{\mathbb{T}}\to\mathbb{R}_{\mathscr{F}}$. *If for* $\forall\,\varepsilon>0,\exists$ *a* Δ*-gauge* δ *s.t. for any partitions* $P=\{[q_{i-1},q_i];\xi_i\}_{i=1}^n$, *which is* δ*-fine*

$$D\Big(\sum_{i=1}^n \tilde{f}(\xi_i)(q_i-q_{i-1})+\tilde{F}(q_{i-1}),\tilde{F}(q_i)\Big)<\varepsilon,$$

then we say $\tilde{f}$ *is fuzzy Henstock-Δ-integrable on* $[a,b]_{\mathbb{T}}$. *We denote* $\tilde{F}(q)=(FH)\int_a^q\tilde{f}(s)\Delta s$, $\tilde{f}\in FH_{[a,b]_{\mathbb{T}}}$.

Definition 3.2 *Suppose* $\exists\,\tilde{f}^{\Delta}(p)\in\mathbb{R}_{\mathscr{F}}$ *and given* $\forall\,\varepsilon>0$, $\exists\,U_{\mathbb{T}}$*(i.e.* $U_{\mathbb{T}}=(p-\iota,p+\iota)\cap[a,b]_{\mathbb{T}}$ *for* $\iota>0$*) s.t.*

$$D\big(\tilde{f}(\sigma(p))\ominus_{gH}\tilde{f}(q),\tilde{f}^{\Delta}(p)(\sigma(p)-q)\big)\le\varepsilon(\sigma(p)-q)\ for\ all\ q\in U_{\mathbb{T}},$$

then $\tilde{f}$ *is fuzzy Δ-differentiable. Denote by* $\tilde{f}^{\Delta}(q)$ *the fuzzy Δ-derivative.*

Definition 3.3 *Suppose* $\tilde{f}:[a,b]_{\mathbb{T}}\to\mathbb{R}_{\mathscr{F}}$. *If for* $\forall\,\varepsilon>0,\exists\,\eta>0$ *s.t. for any disjoint finite interval families* $\{[c_i,d_i]_{\mathbb{T}},1\le i\le n\}$ *satisfy* $\sum_{i=1}^n|d_i-c_i|<\eta$,

$$\sum_{i=1}^n\omega(\tilde{f},[c_i,d_i]_{\mathbb{T}})<\varepsilon,$$

then we say $\tilde{f}$ *is* AC^*.

Definition 3.4 *The family of fuzzy functions is uniformly* ACG^*, *if* $M=\bigcup_{i=1}^n M_i$ *and for* $\forall\,i$, *the family of fuzzy functions is uniformly* AC^*.

Definition 3.8 *Suppose* $\tilde{f}:[a,b]_{\mathbb{T}}\times\mathbb{R}_{\mathscr{F}}\to\mathbb{R}_{\mathscr{F}}$ *belong to the class* $\upsilon([a,b]_{\mathbb{T}}\times\mathbb{R}_{\mathscr{F}},\pi,S)$ *if*

1. $\pi:[a,b]_{\mathbb{T}}\to\mathbb{R}^+$ *is continuous with* $\|\pi\|\le\frac{1}{2U}$, $[a,b]_{\mathbb{T}}\subset[0,U]$.
2. *Every* $S>0$,
$$\{(FH)\int_0^q\tilde{f}(p,\tilde{\vartheta}(p))\Delta p,\|\tilde{\vartheta}\|\le S\}\subset C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$$
is weakly relatively compact and equi-continuous and uniformly ACG^*.
3. *Let every rd-continuous* $\tilde{\vartheta}:[a,b]_{\mathbb{T}}\to\mathbb{R}_{\mathscr{F}},q\in[a,b]_{\mathbb{T}}\to\tilde{f}(q,\tilde{\vartheta}(q))$ *is FH-Δ-integrable and*
$$\limsup_{s\to\infty}\Big(\frac{1}{S}\sup_{\|\tilde{\vartheta}\|\le S}\|\tilde{f}(\cdot,\tilde{\vartheta}(\cdot))\|\Big)<\frac{1}{2}.$$
4. *There is* $\tilde{\vartheta}_n\longrightarrow\tilde{\vartheta}$ *with respect to* $C^*_{rd}([0,1]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ *with* $\tilde{f}(\cdot,\tilde{\vartheta}_n(\cdot))\xrightarrow{weakly}\tilde{f}(\cdot,\tilde{\vartheta}(\cdot))$.

Theorem 3.1 *Suppose* $\tilde{f},\tilde{f}_n\in C_{rd}([0,1]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ *is bounded. Then* $\tilde{f}_n(q)\longrightarrow\tilde{f}(q)$ *if and only if* $\tilde{f}_n(q)\xrightarrow{weakly}\tilde{f}(q)$, $q\in[0,1]_{\mathbb{T}}$.

Theorem 3.2 *Suppose* $\tilde{f}(t),\tilde{f}_n:[a,b]_{\mathbb{T}}\to\mathbb{R}_{\mathscr{F}}$. *If* $\tilde{f}_n$ *is FH-Δ-integrable, let* $\tilde{F}_n$ *is primitive of* $\tilde{f}_n$ *and satisfies:*

1. $\vartheta^*\tilde{f}_n(q)\to\vartheta^*\tilde{f}(q)$ *a.e. for all* $\vartheta^*\in C^*_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$.
2. *The set* $\Pi=\{\vartheta^*\tilde{F}_n,n=1,2,\cdots\}$ *is* ACG^* *uniformly on* $[a,b]_{\mathbb{T}}$, *for* $\forall\,\vartheta^*\in C^*_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$.
3. *The set* Π *is equi-continuous and for* $\forall\,\vartheta^*\in C^*_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$,

then $\tilde{f}\in FH_{[a,b]_{\mathbb{T}}}$ *and*

$$\int_0^q\tilde{f}_n(p)\Delta p\to\int_0^q\tilde{f}(p)\Delta p.$$

Definition 4.1 *If* $\tilde{\vartheta}(q)$ *is a global (i)-solution (or (ii)-solution) of (1) on* $[0,U]$ *iff it is rd-continuous and satisfies:*

$$\begin{cases}\tilde{\vartheta}(q)=\int_0^U\pi(p)\tilde{\vartheta}(p)\Delta p+\int_0^q\tilde{f}(p,\tilde{\vartheta}(p))\Delta p,\\ \tilde{\vartheta}(0)=\int_0^U\pi(p)\tilde{\vartheta}(p)\Delta p\end{cases}\tag{2}$$

or

$$\begin{cases}\int_0^U\pi(p)\tilde{\vartheta}(p)\Delta p=\tilde{\vartheta}(q)+(-1)\int_0^q\tilde{f}(p,\tilde{\vartheta}(p))\Delta p,\\ \tilde{\vartheta}(0)=\int_0^U\pi(p)\tilde{\vartheta}(p)\Delta p\end{cases}\tag{3}$$

respectively.

Theorem 4.1 *Let* $\tilde{f}:[a,b]_{\mathbb{T}}\times\mathbb{R}_{\mathscr{F}}\to\mathbb{R}_{\mathscr{F}}\in\upsilon([a,b]_{\mathbb{T}}\times\mathbb{R}_{\mathscr{F}},\pi,S)$ *and* $[a,b]_{\mathbb{T}}\subset[0,U]$. *Then there is a global (i)-solution and a global (ii)-solution* $\tilde{\vartheta},\tilde{\vartheta}'$ *of (1) for which* $\tilde{\vartheta}(0)=\int_0^U\pi(p)\tilde{\vartheta}(p)\Delta p$.

Conclusions

To popularize the FDEs on time scales, we first propose the notions of FH-Δ-integral and fuzzy Δ-derivative on $[a,b]_T$. Then, we obtain some convergence theorems of the FH-Δ-integral on $[a,b]_T$. Finally, as the application, we obtain an existence theorem of the global (i)-solution and global (ii)-solution for fuzzy differential equations.