D-Beam theory for Functionally Graded Double Cantilever Beam analysis

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Abstract

A formulation that takes advantage of both Layer-Wise and Equivalent Single Layer approaches for modeling Functionally Graded beams is presented. Such alternative formulation, referred to as D-Beam, is here applied to model the Double Cantilever Beam specimen in order to estimate the relevant fracture opening mode of metallic graded beams made up by means of Additive Manufacturing technique. Numerical results are presented.

1 Introduction

Functionally graded (FG) materials are considered of relevant interest in aerospace engineering applications because of their high stiffness-to-weight and strength-to-weight ratios. To build such materials novel approaches, which are alternative with respect to the time-consuming traditional methods, are represented by the Additive Manufacturing (AM) techniques. Among the AM techniques, the direct laser deposition has been used to realize metallic FG material ranging from 100% of Ti6Al4V to 100% of AlSi10Mg to meet the light weight and high strength specification required by industry. Despite the stiffness and hardness properties obtained, cracks were found at layers transition and this evidences the need for fracture mechanics analysis of metallic FG structure made up by AM technologies. The attention is here focused on the modeling and fracture mechanic analysis of FG beam structure. For the 3D modeling of FG beams great modeling and computational effort are generally requested. This motivates the use of 1D beam models but an proper level of modeling accuracy must be ensured. Beam models can belong to the Equivalent Single Layer (ESL) and to the Layer-Wise (LW) models. The first are preferable to reduce the computational burden while the latter to increase the accuracy of the solution. Such approaches can be used to model the Double Cantilever Beam (DCB) specimen allowing to compute the specimen compliance and from this the Energy Release Rate (ERR).

In this work, an alternative approach for laminated composite beams is used (Daví et al. 2014). A layer-wise kinematical model that fulfill the point-wise equilibrium equations, the layer interface continuity conditions and the the traction-free conditions at the top and bottom faces of the beam is obtained. Then, generalized variables that represent the kinematics of the beam as a whole are introduced and, by manipulating the interface continuity conditions, the problem degrees of freedom are reduced obtaining an ESL model that preserves the accuracy of the assumed LW kinematics. The proposed approach is used in combination with the discrete layer technique to solve the FG DCB problem.

2 D-Beam model

The D-Beam model holds for layered linear elastic beams having n layers, rectangular cross-section of height h and width B and length is L. The beam is loaded at its ends by generalized stress resultants. To describe the beam, a global reference system $\{x, z\}$, centered at the left end of the beam, is used. The x-axis lies along the beam midline. Local reference systems $\{x, z_l\}$ are also used to describe the kinematics of each layer l. The thickness of each layer is h_l while the distance between the beam and the layer midlines is $\bar{h}_l$. The LW kinematics is written as

$$\begin{aligned} u_l\left(x, z_l\right) &= u_{0l}\left(x\right) + \theta_l\left(x\right) z_l + B_l\left(x\right) z_l^2 + C_l\left(x\right) z_l^3 \\ w_l\left(x, z_l\right) &= w_{0l}\left(x\right) + A_l\left(x\right) z_l + D_l\left(x\right) z_l^2 \end{aligned} \tag{1}$$

being u_l, w_l and θ_l the axial and transverse displacement and the section rotation of each layer. In Eq. 1, A_l, B_l, C_l and D_l are LW functions of x.

In particular, by using Eq.1 along with linear strain-displacement and linear elastic constitutive relations it is obtained a set of LW ordinary differential equations that are solved by imposing interface continuity conditions (in terms of continuity of displacement and equilibrium between adjacent layers) and taking into consideration the stress free condition of the top and bottom beam surfaces. Such solutions allow to write the LW functions A_l, B_l, C_l and D_l in terms of the layer midline kinematics u_{0l}, w_{0l} and θ_{0l}. A set of useful relationships are also obtained that allow to manipulate the interface displacement continuity conditions in such a way that the LW kinematic variables of each layer u_{0l}, w_{0l} and θ_{0l} are written in terms of the midline kinematics of the specific layer m that contains the beam midline at $z_m = \bar{z}_m$. By introducing the axial U and transverse W displacements of the multilayered beam midline and the beam cross section rotation Θ as generalized variables, namely

$$u_m\left(x, \tilde{z}_m\right) = U, \quad w_m\left(x, \tilde{z}_m\right) = W, \quad \Theta = \left.\frac{\partial u_l}{\partial z_l}\right|_{\tilde{z}_m} \tag{2}$$

the LW kinematical model Eq.1 is then rewritten as

$$\begin{aligned} u_l(x, z) &= U(x) + f_l^{u\Theta}(z)\Theta(x) + f_l^{uU''}(z)U''(x) + f_l^{u\Theta''}\Theta''(x) \\ w_l(x, z) &= W(x) + f_l^{wU'}(z)U'(x) + f_l^{w\Theta'}(z)\Theta'(x) \end{aligned} \tag{3}$$

where $f_l^{\cdot\star}(z)$ are known function of z and of the layers' material constants and primes denote derivative with respect to x. For more detail about the formulation the interested Reader is referred to [?].

Looking at Eq. 3 it appears that the number of unknowns in the present formulation is independent on the number of layers. In addition, as stated before, the continuity and equilibrium conditions at layers' interfaces are met as well as the point-wise balance equations are fulfilled. It follows that the present D-Beam model has the accuracy of the LW models and the computational effectiveness of the ESL ones.

At this point, by using Eq.3 along with the strain-displacement and the layer constitutive equations, it is possible to write the beam constitutive equations as

$$\begin{Bmatrix} N \\ M \\ T \end{Bmatrix} = \begin{bmatrix} k_m & z_G k_m & 0 \\ z_G k_m & k_f & 0 \\ 0 & 0 & \chi^2 k_s \end{bmatrix} \begin{Bmatrix} U' \\ \Theta' \\ \gamma \end{Bmatrix} \tag{4}$$

where N and T are the axial and shearing stress resultant forces, respectively, and M is the bending moment per unit length. In Eq.4, k_m and k_f are the membrane and the flexural stiffness per unit width, respectively, z_G is the position of the beam section elastic centroid with respect to the beam midline, k_s is the shear stiffness per unit width, $\gamma = \Theta + W'$ is the generalized shear strain of the beam and χ^2 is the shear correction factor. The interested reader is referred to [?] for the definitions of all the aforementioned quantities that have been here omitted for the sake of conciseness.

3 Double Cantilever Beam

The proposed D-beam model is used to compute the energy release rate (ERR) of a double cantilever beam (DCB) specimen. It is assumed that the DCB presents a delamination of length a, each arm of the DCB has thickness h_{arm} while the beam width is B. The uncracked part of the DCB is assumed to be perfectly rigid, thus the arms cross-section at the crack tip are modeled as clamped ends. Each DCB's arm is loaded with a tip shearing force F and this results in a mouth opening Δ. The Energy Release Rate (ERR) is the amount of the total potential energy Π that is released as consequence of an infinitesimal advancement of the crack. Taking into account that the material is linear elastic, using the present formulation, it is defined as

$$ERR = \frac{F^2}{B^2}\left[\frac{a^2}{(k_f - k_m z_G^2)} + \frac{1}{\chi^2 k_s}\right] \tag{5}$$

For homogeneous DCB Eq.5 simplifies as

$$ERR = \frac{12F^2a^2}{EB^2h_{arm}^3}(1+f) \tag{6}$$

being E the young modulus along the length direction. The factor f equals zero under the Euler-Bernoulli beam assumption, it is worth $f = Eh_{arm}^2/12\chi^2Ga^2$ under the first order shear deformation Timoshenko's theory and it is worth $f = Eh_{arm}^2/8Ga^2$ using the formulation proposed in this work. In particular, the ERR relationships obtained by the present formulation is equal to one given by (Tada et al. 2000) taking into account the correction due to shear deformation for a isotropic beam having Poisson's ratio $1/3$.

4 Applications

Each arm of the DCB has length $a = 10\,mm$, width $B = 2\,mm$ and thickness $h_{arm} = 0.25\,mm$. It undergoes a shearing resultant force at the free end $T_0 = 1\,N/m$. The DCB's arm are assumed to be realized by functionally graded material with Ti6Al4V and AlSi10Mg as constituent phases. Material properties of Ti6Al4V $E = 110\,GPa$, $G = 40\,GPa$ and $\nu_z = 0.31$ while material properties of AlSi10Mg are $E = 68\,GPa$, $G = 26.5\,GPa$ and $\nu_z = 0.33$. The grading law for the upper arm is $V_B = \frac{(\bar{h}_l + z_l + h_{arm}/2)^\eta}{h_{arm}^\eta}$ and $V_A = 1 - V_B$ and the Young's and Shear moduli and Poisson's ratio are computed by the rule of mixtures. The DCB lower arm is symmetric to the upper one with respect to the DCB mean-line where the crack is assumed to lie. The exponent η in in the grading law is the grading index and it is let vary as $0 \leq \eta \leq 2$.

To set the number of discrete layer that allows the appropriate modeling of the FG beam, a convergence analysis on the tip deflection, cross section rotation and ERR is carried for different grading index η. Results are collected in Table 1 along with computation time showing that 100 layers are enough to gain converging results but 1000 layers can be used considered the very low computational requirements of the method.

Table 1: Discrete layer approximation convergence for $\eta = 0.5$

Numb. of Layers	10	50	100	500	1000
$W \times 10^{-6}$ [m]	3.115	3.107	3.107	3.107	3.107
$\Theta \times 10^{-4}$ [rad]	4.669	4.658	4.657	4.657	4.657
ERR [Pa m]	233.5	232.9	232.9	232.9	232.9
CPU$-time$ [s]	0.039	0.041	0.058	0.112	0.369

The through the thickness distribution of the FG material properties are shown in Figure 1 (a) and (b) for two different grading index, the number of layers is set to 1000 when the Ti6Al4V is at the DCB mean-line.

Figure 1(c) reports the value of the ERR computed by using the proposed formulation as function of the grading index η for two different phase arrangements. In particular, the continuous line of Figure 1(c) shows the trend of the ERR versus the grading index when the Ti6Al4V is assumed to lay at the DCB mean-line while the AlSi10Mg is at top and bottom surface of the DCB's arms, namely the Ti6Al4V is the phase A. It appears that as the grading index increases, i.e. when the presence of the more stiffer phase is greater across the DCB mean-line where is the crack, the ERR decreases. The opposite trend is obtained for the DCB realized with the AlSI10Mg on the mean-line and Ti6Al4V on the top and bottom surfaces as shown by the dashed line in Figure 1 (c).

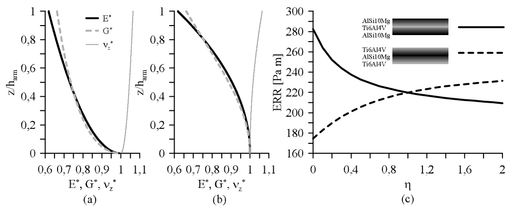

Figure 1: Through the thickness distribution of material constants ($E^* = E/E_{Ti6Al4V}$, $G^* = G/G_{Ti6Al4V}, \nu_z^* = \nu_z/\nu_{Ti6Al4V}$) for (a) $\eta = 0.5$ and (b) $\eta = 2$ in the upper arm of the DCB, and (c) EER as function of η

5 Conclusions

An alternative beam formulation, called D-Beam, has been presented in this work for the analysis of DCB FG specimen. The D-Beam approach uses a LW kinematics that fulfill the point-wise equilibrium equations as well as the interface continuity conditions. The LW relations are then lumped into an ESL model by introducing generalized kinematic variables. The FG material properties have been modeled by means of discrete layer technique and the proposed D-Beam approach has been employed to study the dependency of the strain ERR as function of the grading law that can be obtained through AM process.

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Short reference list

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